MATH3705 Tutorial 2

1. Calculate the Laplace transform of

$$f(t) = \sinh(at) = \frac{e^{at} - e^{-at}}{2}$$

Solution:

$$\begin{aligned}
\mathcal{L}(\sinh(at)) &= \mathcal{L}\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at}) \\
&= \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a} \\
&= \frac{1}{2}\frac{s+a-(s-a)}{s^2-a^2} = \frac{a}{s^2-a^2}
\end{aligned}$$

2. Using the shift theorem find the Laplace transform of

$$f(t) = e^{2t}t^2$$

Solution: Recall the first shift theorem says

$$\mathcal{L}\left(e^{-at}f(t)\right) = F(s-a)$$

where $\mathcal{L}(f) = F(s)$. Now, we know that

$$\mathcal{L}\left(t^2\right) = \frac{2!}{s^3} = \frac{2}{s^3}$$

so, by the shift theorem

$$\mathcal{L}\left(e^{2t}t^2\right) = \frac{2}{(s-2)^3}$$

3. Using the Laplace transform solve the differential equation

$$f'' + f' - 6f = e^{-3t}$$

with boundary conditions $f(0) = f'(0) = 0$.

Solution: The subsidiary equation is

$$s^2F + sF - 6F = \frac{1}{s+3}$$

or

$$F = \frac{1}{(s+3)^2(s-2)}$$

As before, we do partial fractions

$$\begin{aligned}
\frac{1}{(s+3)^2(s-2)} &= \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s-2} \\
1 &= A(s+3)(s-2) + B(s-2) + C(s+3)^2
\end{aligned}$$

$s=-3$ gives $B=-1/5$ and $s=2$ gives $C=1/25$. Putting in $s=1$ we find

$$1=-4A+\frac{1}{5}+\frac{16}{25}$$

and so $A=-1/25$. Putting all this together says that

$$f=-\frac{1}{25}e^{-3t}-\frac{t}{5}e^{-3t}+\frac{1}{25}e^{2t}$$

4. Find $L\left\{f(t)\right\}$, where $f(t)=\sqrt{t}+2t^3-e^{-3t}\cos(4t)$.

   Solution: By the First Shift Theorem, we have

$$L\{e^{-3t}\cos(4t)\}=[L\{\cos(4t)\}]\,|_{t+3}=\left[L\frac{s}{s^2+16}\right]|_{s+3}=\frac{(s+3)}{(s+3)^2+16}.$$

   By linearity of LT, we have

$$F(s)=\frac{\sqrt{\pi}}{2s^{3/2}}+\frac{12}{s^4}-\frac{(s+3)}{(s+3)^2+16}.$$

5. Find $L\left\{f(t)\right\}$, where $f(t)=\begin{cases} t, & 0\le t<2; \\ e^{2t}\sin 3(t-2), & t\ge 2. \end{cases}$

   Solution: We write $f(t)$ as

$$f(t)=[u(t)-u(t-2)]t+u(t-2)e^{2t}\sin 3(t-2).$$

   Then

$$F(s)=\frac{1}{s^2}+e^{-2s}\left[\frac{3e^4}{(s-2)^2+9}-\frac{1}{s^2}-\frac{2}{s}\right].$$

6. Find $L^{-1}\left\{\frac{e^{-3s}}{s(s^2+2s+2)}\right\}$.

   Solution: Let $G(s)=\frac{1}{s(s^2+2s+2)}$. By partial fraction, we have

$$G(s)=\frac{\frac{1}{2}}{s}-\frac{\frac{1}{2}s+1}{s^2+2s+2}=\frac{1}{2}\left(\frac{1}{s}\right)-\frac{1}{2}\left(\frac{s+1}{(s+1)^2+1}\right)-\frac{1}{2}\left(\frac{1}{(s+1)^2+1}\right).$$

   Thus

$$g(t)=\frac{1}{2}-\frac{1}{2}e^{-t}\cos(t)-\frac{1}{2}e^{-t}\sin(t).$$

   By the Second Shift Theorem,

$$f(t)=u(t-3)\left[\frac{1}{2}-\frac{1}{2}e^{-(t-3)}\cos(t-3)-\frac{1}{2}e^{-(t-3)}\sin(t-3)\right].$$